

Bounds on Block Error Probability for Multilevel Concatenated Codes

Hari T. Moorthy, Diana Stojanovic and Shu Lin¹

Department of Electrical Engineering,
University of Hawaii,
Honolulu, Hawaii 96822

Email: harit@spectra.eng.hawaii.edu
dixi@spectra.eng.hawaii.edu

Abstract — Maximum likelihood decoding of long block codes is not feasible due to large complexity. Some classes of codes are shown to be decomposable into multilevel concatenated codes (MLCC) [3]. For these codes, multistage decoding provides good trade-off between performance and complexity. In this paper, we derive an upper bound on the probability of block error for MLCC. We use this bound to evaluate difference in performance for different decompositions of some codes. Examples given show that a significant reduction in complexity can be achieved when increasing number of stages of decoding. Resulting performance degradation varies for different decompositions. A guideline is given for finding good m-level decompositions.

I. INTRODUCTION

Multistage Decoding (MSD) is an efficient soft decision decoding method for long decomposable codes, such as multilevel concatenated codes (MLCC) [1]. Although suboptimum in performance, it greatly reduces the computational complexity as compared to optimum decoding.

Since its inception in Hemmati's paper [2], Closest Coset Decoding (CCD) of $|u|u + v|$ codes and generalizations of CCD have been investigated by several authors. However, only a few papers on the performance analysis of this method have appeared [3-6]. Furthermore, these methods are restricted to $|u|u + v|$ codes and some others to block modulation codes. The different code-structure of MLCC's precludes application of the analysis therein to the case of MLCCs.

In [3], an upper bound on the effective error coefficient (EEC) for 2-stage decoding of MLCC's was derived and some guidelines for choosing a good 2-level decomposition of Reed Muller codes were given.

In this paper, we first derive an upper bound on block error probability of MSD of MLCC's, when optimum decoding of each stage is performed. We first express the upper bound in terms of all the error coefficients, and then explain how these coefficients can be obtained using some combinatorial methods and weight

distributions of inner and outer codes of a MLCC. The bound enables prediction of the performance of MSD (two or more levels) without simulation. Therefore an estimate of performance degradation, when the number of decoding stages is increased, can be obtained.

II. ANALYSIS OF CCD OF MULTILEVEL CONCATENATED CODES

For an M -level code $C = \{B_1, B_2, \dots, B_M\} * \{A_1, A_2, \dots, A_M\}$, consider the i -th stage of decoding for $1 \leq i \leq M$. At each stage, decoding is assumed to be complete (an estimate of a transmitted codeword is always given) soft decision maximum likelihood closest coset decoding [5].

Let $P_e^{(i)}$ denote the probability that the i -th stage decoder makes an error when all previous $(i-1)$ stages are correct. Then, conditioning on the i -th stage decoding being in error, we obtain $P_e = \sum_{i=1}^M P_e^{(i)}$. In general, $P_e^{(i)}$ depends on the codeword at the output of the i -th inner encoder. However, for binary linear codes, BPSK transmission and the AWGN channel, $P_e^{(i)}$ is the same for all transmitted codewords [7]. For simplicity, we assume that the all-zero codeword is transmitted. In this case, the output of each B_i -encoder at the transmitter is the all-zero codeword in B_i . The received $(n_A n_B)$ -tuple $\mathbf{r} = (r^{(1)}, r^{(2)}, \dots, r^{(n_B)})$ is sectionalized into n_B sections each of length n_A , where n_B and n_A are lengths of outer and inner code respectively.

An error at the i -th stage, when CCD is used, occurs if a nonzero codeword $\mathbf{b} = (b_1, b_2, \dots, b_{n_B}) \in B_i$ has larger correlation than the all-zero codeword $\mathbf{z} \in B_i$. Let $\mathbf{a}^j = (a_1^j, a_2^j, \dots, a_{n_A}^j)$ be the codeword with the best correlation in the coset corresponding to j -th symbol b_j , for $j = 1, 2, \dots, n_B$. If a standard BPSK mapping ($0 \rightarrow 1, 1 \rightarrow -1$) is applied, then the difference in metrics of \mathbf{b} and \mathbf{z} can be bounded by the following expression:

$$M(\mathbf{b}) - M(\mathbf{z}) \leq \sum_{j=1, b_j \neq 0}^{n_B} \sum_{l=1, a_l^j \neq 0}^{n_A} (-2\tau_l^j), \quad (1)$$

where $M(\cdot)$ denotes correlation metric. Thus the probability of error at the i -th stage is given by

$$P_e^{(i)} = P(M(\mathbf{b}) - M(\mathbf{z}) > 0, \mathbf{b} \in B_i, \mathbf{b} \neq \mathbf{0}). \quad (2)$$

¹This work was supported by NSF grant NCR-9415374 and NASA grant NAG 5-931.

For a particular \mathbf{a} and \mathbf{b} ,

$$T(\mathbf{a}, \mathbf{b}) = \sum_{j=1, b_j \neq 0}^{n_B} \sum_{l=1, a_l^j = 1}^{n_A} (-2r_l^j)$$

is a Gaussian random variable with mean μ and variance σ^2 given by

$$\mu = 2A \sum_{b_j \neq 0} w(\mathbf{a}), \quad \sigma^2 = 4 \sum_{b_j \neq 0} w(\mathbf{a})$$

where A is the amplitude of a transmitted BPSK signal (SNR=10 log $(A^2/2 k/n)$). It then follows that

$$\begin{aligned} P^{(i)} &\leq \sum_{\mathbf{b} \in \mathbf{B}_i} \sum_{b_j \neq 0} \sum_{\mathbf{a} \in \Omega(b_j)} P(T(\mathbf{a}, \mathbf{b}) > 0) \\ &= \sum_{\mathbf{b} \in \mathbf{B}_i} \sum_{b_j \neq 0} \sum_{\mathbf{a} \in \Omega(b_j)} \frac{1}{2} \operatorname{erfc} \left(A \sqrt{\frac{1}{2} \sum_{b_j \neq 0} w(\mathbf{a})} \right), \quad (3) \end{aligned}$$

where $\Omega(b_j)$ represents the coset corresponding to the symbol b_j , and $w(\cdot)$ the weight of a codeword.

Note that for any $0 \neq \mathbf{b} \in \mathbf{B}_i$, $0 \neq \mathbf{a} \in \mathbf{A}_i$, $\sum_{b_j \neq 0} w(\mathbf{a}) \geq d_{\min}(\mathbf{C})$

Definition 1: Define $S_i(Y)$ as the number of terms of the form $\frac{1}{2} \operatorname{erfc} \left(A \sqrt{Y/2} \right)$ satisfying

1. $Y = \sum_{b_j \neq 0} w(\mathbf{a})$
2. $\mathbf{b} \in \mathbf{B}_i, \mathbf{a} \in \Omega(b_j)$

The union bound on the probability of error at the i -th stage of decoding is then given by

$$P_e^{(i)} < \sum_Y S_i(Y) \frac{1}{2} \operatorname{erfc} \left(A \sqrt{Y/2} \right) \quad (4)$$

where the summation is taken over all Y defined above. Thus

$$P_e < \sum_i \sum_Y S_i(Y) \frac{1}{2} \operatorname{erfc} \left(A \sqrt{Y/2} \right). \quad (5)$$

$S_i(Y)$ depends on the symbol weight distribution of the i -th outer code \mathbf{B}_i , the Hamming weight distribution of the MLC code \mathbf{C} and critically on the weight distribution (of n_A -tuples) in the cosets of the partition $\mathbf{A}_i/\mathbf{A}_{i+1}$. For inner codes of small lengths the coset weight distribution can be evaluated easily by an exhaustive computer search program or using combinatorial methods.

III. RESULTS FOR SOME MLCC CODES

Consider the decomposition

$$\begin{aligned} \text{RM}(64, 42, 16) &= \{(8, 1, 8)(8, 4, 4)^3, (8, 7, 2)^3(8, 8, 1)\} * \\ &\quad \{(8, 8, 1), (8, 4, 4), (8, 0)\} \end{aligned}$$

given in [3]. The symbol weight distribution of \mathbf{B}_1 is $N_{B_1}(0) = 1, N_{B_1}(4) = 98, N_{B_1}(6) = 1, 176, N_{B_1}(7) = 1, 344, N_{B_1}(8) = 5, 573$. The symbol weight distribution of \mathbf{B}_2 is $N_{B_2}(0) = 1, N_{B_2}(1) = 8, N_{B_2}(2) = 812, N_{B_2}(3) = 23, 576, N_{B_2}(4) = 443, 030, N_{B_2}(5) = 5, 315, 576, N_{B_2}(6) = 39, 867, 212, N_{B_2}(7) = 170, 859, 368, N_{B_2}(8) = 320, 361, 329$. The coset weight distributions of $\mathbf{A}_1/\mathbf{A}_2$ and $\mathbf{A}_2/\{0\}$ are given in Table 1.

To compute $S_i(Y)$ with $i = 1, 2$, we have to determine how many $n_A n_B$ -tuples of weight Y can result from a $\mathbf{b} \in \mathbf{B}_1$ of symbol weight $w_1(\mathbf{B}_1) = X$. In order to answer this question, we must see if a partition of the integer Y into X parts is possible with each part belonging to a valid entry in the coset weight distribution of the partition $(8, 8, 1)/(8, 4, 4)$. A partition of the form $8 = 2 + 3 + 1 + 2$ is invalid because none of the non-zero cosets in the above partition have a vector of Hamming weight 1 or 3. Furthermore, the partitions must be counted in an ordered fashion (i.e., we are counting the so called distributions of an integer). For example, there are 12 distributions of 16 into parts $\{4, 4, 2, 6\}$. The partition table for first two nonzero weights in both first and second stage are given in Table 2.

The effective error coefficient in stage-1 is determined as $S_1(8) = 98 \times 256 = 25,088$. The next two error coefficients are $S_1(10) = 98 \times 4 \times 4^3 \times 8 = 200,704$ and $S_1(12) = 98 \times (6 \times 8^2 \times 4^2 + 4^4) + 4^6 \times 1176 = 544,096$. The higher order error coefficients can be computed in a similar manner.

In the second stage of decoding, the effective error coefficient is $S_2(8) = 8 + 784 = 792$ and the error coefficient $S_2(12) = 18,816$. The first three error coefficients for this 2-level decomposition of RM $(64, 42, 8)$ corresponding to weights 8, 10 and 12 are $S_1(8) + S_2(8) = 25,880$, $S_1(10) = 200,704$ and $S_1(12) + S_2(12) = 5,462,912$.

In summary, for the Reed-Muller code $\mathbf{C} = (64, 42, 8)$, an upper bound (with first two significant terms) on the block error probability with two-stage decoding is

$$\begin{aligned} P_{e,2\text{-stage}} &< 2.5 \cdot 10^4 \frac{1}{2} \operatorname{erfc} \left(A \sqrt{8/2} \right) + \\ &2.0 \cdot 10^5 \frac{1}{2} \operatorname{erfc} \left(A \sqrt{10/2} \right) + 5.462,912 \frac{1}{2} \operatorname{erfc} \left(A \sqrt{12/2} \right). \end{aligned}$$

The complexity of stage-2 can be reduced by employing a 3-stage decomposition

$$\begin{aligned} \text{RM}(64, 42, 8) &= \\ &\{(8, 1, 8)(8, 4, 4)^3, (8, 7, 2)^3, (8, 7, 2)(8, 8, 1)\} * \\ &\{(8, 8, 1), (8, 4, 4), (8, 2, 4), (8, 0)\} \quad (6) \end{aligned}$$

for which

$$P_{e,3-stage} < 2.6 \cdot 10^4 \frac{1}{2} \operatorname{erfc} \left(A\sqrt{8/2} \right) + 2.0 \cdot 10^5 \frac{1}{2} \operatorname{erfc} \left(A\sqrt{10/2} \right) + 5.4 \cdot 10^6 \frac{1}{2} \operatorname{erfc} \left(A\sqrt{12/2} \right).$$

Notice that the asymptotic loss in decoding gain, due to this simplification, predicted from the bound is practically zero, and this agrees with actual simulation results shown in Figure 1.

Table 1

Coset-#s	Weight of vector				
	0	2	4	6	8
0	1	-	14	-	1
1-7	-	4	8	4	-

Table 2

$w_s(b)$	Partition	N_1	N_2
4	{2, 2, 4, 4}	6	$4^2 8^2 6$
4	{2, 2, 2, 6}	4	$4^3 4$
4	{2, 2, 2, 4}	4	$4^3 8$
5	Not possible	-	-
7	Not possible	-	-
6	{2, 2, 2, 2, 2, 2}	1	4^6

N_1 = Number of partitions. N_2 = Number of code-words in $B_i * [A_i/A_{i+1}]$ per partition.

For long codes, even 2-stage CCD cannot be performed in an optimal manner due to the complexity of the trellises involved. Hence it is imperative that tight bounds be derived so that the loss of coding gain at high SNRs due to m -stage ($m \geq 3$) can be predicted.

The loss in coding gain of 2-stage relative to m -stage decoding can be determined based on the error coefficients derived in the paper. For the code $C = (128, 64, 16)$ RM, and $n = 2, 3$, this is shown in Figure 2.

The corresponding expressions are

$$P_{e,2-stage} < 13.8 \cdot 10^6 \frac{1}{2} \operatorname{erfc} \left(A\sqrt{16/2} \right) + 5.1 \cdot 10^9 \frac{1}{2} \operatorname{erfc} \left(A\sqrt{24/2} \right)$$

$$P_{e,3-stage} < 1.6 \times 10^9 \frac{1}{2} \operatorname{erfc} \left(A\sqrt{16/2} \right) + 6.1 \times 10^{13} \frac{1}{2} \operatorname{erfc} \left(A\sqrt{24/2} \right)$$

The three stage decomposition has a trellis complexity of 512, 128, and 512 states for the first, second, and third stage, and the average number of operations per information bit is 1263. The most of the computational complexity is due to a large and fully connected trellis of the third stage, so a way to reduce it is to implement 4-stage decoding by dividing the third stage into two stages.

The difference between the bounds for 2-stage and 3-stage at block error rate 10^{-6} is about 0.6 dB. Since the simulation of 2-stage decoding with MLD at each stage is not available, we compared the results of 3-stage decoding with a 2-stage decoding where the first stage is decoded using iterative search. We believe this would be very close in performance of optimum 2-stage decoding. The difference in performance for this 2-stage and optimum 3-stage decoding is 0.6 dB as well.

The third example is done for the (72, 52, 8) decomposable code. Bounds for 3 and 4-stage decoding of this code are given in Figure 3.

The state complexities are 2, 512, 8, 1 for the 4-stage decoding, and {1024, 8, 1} for the 3-stage decoding.

IV. TRADE-OFFS BETWEEN COMPLEXITY AND PERFORMANCE

We have shown in the previous section that the performance of 2 and 3-stage decoding of (64, 42, 8) code are almost identical. However, this is true only for 3-stage decomposition (6). Reduction in state complexity is not significant, since maximum number of states in the first stage remains 128 (the same as for 2-stage decomposition) and the maximum numbers of states at the second and third stage are 4 and 2 respectively (as compared to 8 states in 2-stage decomposition). The average number of operations per information bit, though, is reduced from 393 to 133 (almost three times).

For another 3-stage decomposition,

$$\begin{aligned} \text{RM}(64, 42, 16) = \\ \{(8, 1, 8)(8, 4, 4), (8, 4, 4)^2, (8, 7, 2)^3(8, 8, 1)\} * \\ \{(8, 8, 1), (8, 6, 4), (8, 4, 4), (8, 0)\}, \end{aligned} \quad (7)$$

maximum number of states is reduced to 16 (in the second stage), so 8 times compared to two stage decomposition. Unfortunately, this decomposition loses about 0.5 dB in performance, and the average number of operations per information bit is 227. Thus the 3-stage decomposition (6) outperforms decomposition (7), while having smaller computational complexity.

We can see that the performance of the multistage scheme depends mainly on the first decoding stage. This suggests that when increasing the number of stages, lower level stages should be divided, while the first should be kept as long as the number of states in the outer code trellis is not too big. In such manner, the performance degradation will not be large (in some cases almost negligible). Since lower level codes in multilevel decomposition are high rate codes with complex trellis structure, splitting each of them into two levels will reduce the complexity significantly (as we have seen in the example of RM(64, 42, 8) code). This can be used to determine a good decomposition of a code.

V. CONCLUSION

In this paper, we have shown how to bound the performance of multistage decoding of Multilevel concatenated codes. We presented examples which show that the simulation results converge to derived bounds at moderate to high signal-to-noise ratios. Therefore, an estimate of the difference in performance for different decompositions and different numbers of decoding stages can be determined. Analysis of reduction in complexity when using large number of decoding stages has also been discussed.

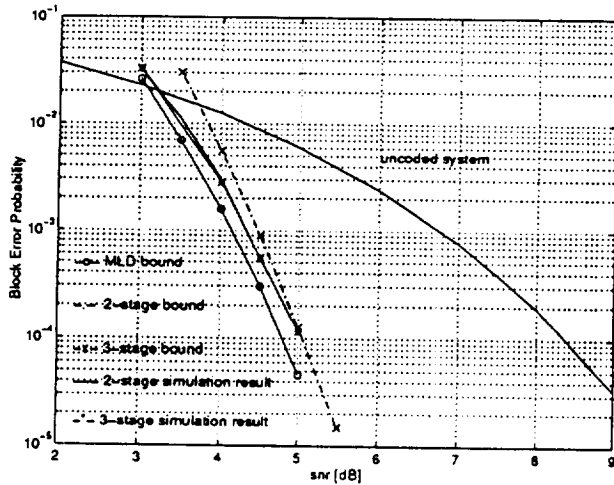


Figure 1: Simulation results and performance bounds for RM(64,42,8) code

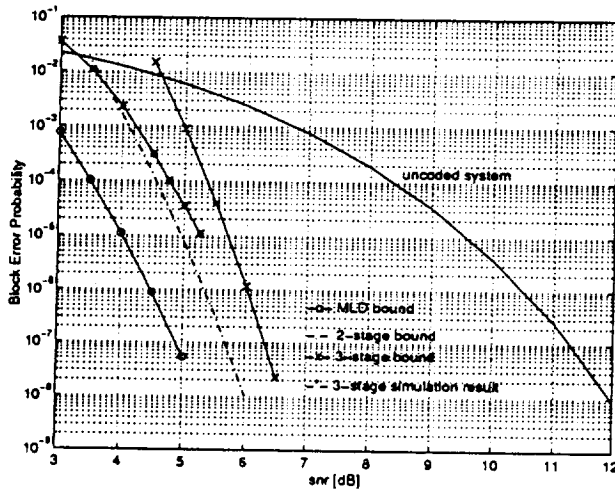


Figure 2: Simulation results and performance bounds for RM(128,64,16) code

REFERENCES

- [1] V. A. Zinoviev, "Generalized Concatenated Codes," *Problemy Peredachi Informatsii*, Vol. 12., No. 1, pp. 5-15, 1976.
- [2] F. Hemmati, "Closest Coset Decoding of $[u|u+v]$ codes," *IEEE J. Selected Areas Communications*, Vol. 7, pp. 982-988, Aug. 1989.
- [3] J. Wu and S. Lin, "An Upper Bound on Effective Error Coefficient of Two-Stage Decoding, and Good Two-Level Decompositions of Some Reed-Muller Codes," *IEEE Transactions on Communications*, Vol. 42, No. 3, March 1994.
- [4] M.P.C. Fossorier and S. Lin, "Weight Distribution for Closest Coset Decoding of $[u|u+v]$ Constructed Codes," *submitted to IEEE Transactions on Information Theory*, in revision. November 1995.
- [5] T. Takata et. al., "An Improved Union Bound on Block Error Probability for Closest Coset Decoding," *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, Vol. E78-A, No. 9, September 1995.
- [6] T. Takata, Y. Yamashita, T. Fujiwara, T. Kasami and S. Lin, "Suboptimum Decoding of Decomposable Block Codes," *IEEE Transactions on Information Theory*, Vol. 40, No. 5, Sep. 1994.
- [7] T. Takata, S. Ujita, T. Kasami and S. Lin, "Multistage Decoding of Multilevel Block M-PSK Modulation Codes and Its Performance Analysis," *IEEE Transactions on Information Theory*, Vol. 39, No. 4, July 1993.

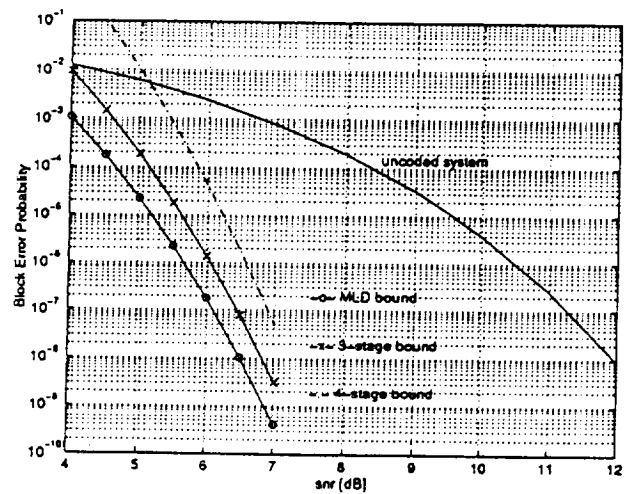


Figure 3: Performance bounds for (72,52,8) decomposable code